Uppgift 6.24b

Vi skall undersöka följande generaliserade integral:

$$\int_1^\infty \frac{\ln(x+1)}{x^3}\,dx.$$

Lösning: En standardkalkyl ger

$$\int_1^b \frac{\ln(x+1)}{x^3}\,dx = \int_1^b \left(\underbrace{x^{-3}}_{\uparrow} \cdot \underbrace{\ln(x+1)}_{\downarrow}\right) dx = \left[-\frac{1}{2x^2}\ln(x+1)\right]_1^b - \int_1^b \left(\left(-\frac{1}{2x^2}\right)\cdot\frac{1}{x+1}\right) dx =$$
$$= -\frac{1}{2}\left[\frac{1}{x^2}\ln(x+1)\right]_1^b + \frac{1}{2}\int_1^b \left(\frac{1}{x^2}\cdot\frac{1}{x+1}\right) dx =$$
$$= -\frac{1}{2}\left[\frac{1}{x^2}\ln(x+1)\right]_1^b + \frac{1}{2}\int_1^b \left(\frac{1}{x^2} + \frac{1}{x+1} - \frac{1}{x}\right) dx =$$
$$= -\frac{1}{2}\left[\frac{1}{x^2}\ln(x+1)\right]_1^b + \frac{1}{2}\left[-\frac{1}{x} + \ln(x+1) - \ln x\right]_1^b =$$
$$= -\frac{1}{2}\left[\frac{1}{b^2}\ln(b+1) - \ln 2\right] + \frac{1}{2}\left[-\frac{1}{b} + \ln(b+1) - \ln b + 1 - \ln 2\right] =$$
$$= \frac{1}{2}\left[\ln 2 - \frac{1}{b^2}\ln(b+1) - \frac{1}{b} + \ln(b+1) - \ln b + 1 - \ln 2\right] =$$
$$= \frac{1}{2}\left[-\frac{1}{b^2}\ln(b+1) - \frac{1}{b} + \ln(b+1) - \ln b + 1\right] =$$
$$= \frac{1}{2}\left[-\frac{1}{b^2}\ln(b+1) - \frac{1}{b} + \ln\frac{b+1}{b} + 1\right] \to \frac{1}{2}\cdot 1 = \frac{1}{2}$$

eftersom

$$\frac{1}{b^2}\ln(b+1) \to 0, \qquad \frac{1}{b} \to 0, \qquad \ln\frac{b+1}{b} = \ln\left(1+\frac{1}{b}\right) \to \ln 1 = 0$$

då $b \to \infty$.

Svar: $\int_1^\infty \frac{\ln(x+1)}{x^3}\,dx = \frac{1}{2}$.