Uppgift 2.34

Vi skall lösa den partiella differentialekvationen

$$x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = x$$

i första kvadranten, och där z åtminstone är av klass C^2 där. För att klara av detta byter vi till de lämpliga koordinaterna

$$\begin{cases} u = 2xy \\ v = 1/y. \end{cases}$$

Vi får då

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = 2y\frac{\partial z}{\partial u} = \frac{2}{v}\frac{\partial z}{\partial u}, \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = 2x\frac{\partial z}{\partial u} - \frac{1}{y^2}\frac{\partial z}{\partial v} = uv\frac{\partial z}{\partial u} - v^2\frac{\partial z}{\partial v} \\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{2}{v}\frac{\partial}{\partial u}\right)\left(\frac{2}{v}\frac{\partial z}{\partial u}\right) = \frac{4}{v^2}\frac{\partial^2 z}{\partial u^2} \quad \text{samt} \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \left(uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}\right)\left(\frac{2}{v}\frac{\partial z}{\partial u}\right) = 2u\frac{\partial^2 z}{\partial u^2} - 2v^2\left[-\frac{1}{v^2}\frac{\partial z}{\partial u} + \frac{1}{v}\frac{\partial^2 z}{\partial u \partial v}\right] = \\
&= 2u\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial z}{\partial u} - 2v\frac{\partial^2 z}{\partial u \partial v}
\end{aligned}$$

så att ekvationen lyder

$$\frac{1}{2}uv\frac{4}{v^2}\frac{\partial^2 z}{\partial u^2} - \frac{1}{v}\left(2u\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial z}{\partial u} - 2v\frac{\partial^2 z}{\partial u \partial v}\right) + \frac{2}{v}\frac{\partial z}{\partial u} = \frac{1}{2}uv$$

eller

$$2z''_{uv} = \frac{1}{2}uv.$$

Men

$$z''_{uv} = \frac{1}{4}uv \Rightarrow z'_u = \frac{1}{8}uv^2 + f(u) \Rightarrow z = \frac{1}{16}u^2v^2 + F(u) + g(v)$$

där F är en primitiv till f. I ursprungliga koordinaterna heter detta

$$z(x,y) = \frac{1}{4}x^2 + F(2xy) + g(1/y)$$

eller

$$z(x,y) = \frac{1}{4}x^2 + a(xy) + b(y)$$

om vi sätter $a(t) \coloneqq F(2t)$ och $b(t) \coloneqq g(1/t)$.