Variabelbyte i partiell differentialekvation

Vi skall bestämma alla C^2-lösningar $z(x,y)$ till differentialekvationen

$$xz''_{xx} - yz''_{xy} + z'_x = 2xy, \qquad x > 0, y > 0$$

genom att byta till nya oberoende variabler (u,v) enligt

$$\begin{aligned} u &= xy \\ v &= y \end{aligned}$$

med inversen

$$\begin{aligned} x &= \frac{u}{v} \\ y &= v. \end{aligned}$$

Lösning: Vi översätter derivatorna:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} = v\frac{\partial z}{\partial u},$$

Det finns två sätt att ta fram $\partial^2 z/\partial x^2$. Det ena sättet är att notera att *operatorn*

$$\frac{\partial}{\partial x} = v\frac{\partial}{\partial u},$$

som ger

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = v\frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right) = v^2\frac{\partial^2 z}{\partial u^2}$$

där vi använt regeln för derivering av en produkt ($v \cdot \partial z/\partial u$) och noterar att $\partial v/\partial u = 0$ eftersom *partiell* derivering m.a.p. u innebär att man håller de andra variablerna (här bara v) *konstanta*. Det andra sättet att ta fram $\partial^2 z/\partial x^2$ är att använda kedjeregeln igen:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(v\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(v\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial x} = \frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right)\cdot y + \frac{\partial}{\partial v}\left(v\frac{\partial z}{\partial u}\right)\cdot 0 = \\ &= v\frac{\partial^2 z}{\partial u^2}\cdot y = v\frac{\partial^2 z}{\partial u^2}\cdot v = v^2\frac{\partial^2 z}{\partial u^2}. \end{aligned}$$

Samma sak igen! Då återstår bara

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(v\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(v\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial y} = \\ &= \frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right)\cdot x + \frac{\partial}{\partial v}\left(v\frac{\partial z}{\partial u}\right)\cdot 1 = v\frac{\partial^2 z}{\partial u^2}\cdot\frac{u}{v} + \frac{\partial z}{\partial u} + v\frac{\partial^2 z}{\partial u \partial v} = \\ &= u\frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + v\frac{\partial^2 z}{\partial u \partial v} \end{aligned}$$

där vi använt kedjeregeln och produktregeln. I de nya variablerna blir ekvationen sålunda

$$\frac{u}{v}\; v^2 \frac{\partial^2 z}{\partial u^2} - v\left(u \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + v \frac{\partial^2 z}{\partial u \partial v}\right) + v \frac{\partial z}{\partial u} = 2u$$

som förenklas till

$$-v^2 \frac{\partial^2 z}{\partial u \partial v} = 2u,$$

d.v.s.

$$z''_{uv} = -\frac{2u}{v^2} \Leftrightarrow z'_u = \frac{2u}{v} + f(u) \Leftrightarrow z(u, v) = \frac{u^2}{v} + F(u) + g(v).$$

Slutligen översätter vi lösningen till de gamla variablerna:

$$z(x, y) = x^2 y + F(xy) + g(y).$$