Uppgift A7.6

Vi skall beräkna

$$\lim_{x\to\infty} x\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2+2x-3}\right).$$

Lösning:

$$x\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2+2x-3}\right) = x\left(\sqrt[3]{x^3\left(1+\frac{3}{x}\right)}-\sqrt{x^2\left(1+\frac{2}{x}-\frac{3}{x^2}\right)}\right) =$$
$$= x^2\left(\sqrt[3]{1+\frac{3}{x}}-\sqrt{1+\frac{2}{x}-\frac{3}{x^2}}\right) = \begin{bmatrix} t=\frac{1}{x} \\ t\to 0^+ \end{bmatrix} = \frac{\sqrt[3]{1+3t}-\sqrt{1+2t-3t^2}}{t^2}.$$

Vi använder nu standardutvecklingen

$$(1+\xi)^\alpha = 1+\alpha\xi+\frac{\alpha(\alpha-1)}{2!}\xi^2+\mathcal{O}(\xi^3).$$

Sålunda

$$(1+\xi)^{1/3} = 1+\frac{1}{3}\xi-\frac{1}{9}\xi^2+\ \mathcal{O}(\xi^3)$$
$$(1+3t)^{1/3} = 1+t-t^2+\ \mathcal{O}(t^3)$$

och

$$(1+\xi)^{1/2} = 1+\frac{1}{2}\xi-\frac{1}{8}\xi^2+\mathcal{O}(\xi^3)$$
$$(1+2t-3t^2)^{1/2} = 1+\frac{1}{2}(2t-3t^2)-\frac{1}{8}(2t-3t^2)^2+\mathcal{O}(t^3) =$$
$$= 1+t-2t^2+\mathcal{O}(t^3).$$

Därför

$$\frac{\sqrt[3]{1+3t}-\sqrt{1+2t-3t^2}}{t^2} = \frac{1+t-t^2+\ \mathcal{O}(t^3)-\left(1+t-2t^2+\mathcal{O}(t^3)\right)}{t^2} = \frac{t^2+\ \mathcal{O}(t^3)}{t^2} =$$
$$= 1+\mathcal{O}(t)\to 1$$

då $t\to 0$.

Svar:

$$\lim_{x\to\infty} x\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2+2x-3}\right) = 1.$$