Uppgift 2.38

Låt u vara en funktion på planet. I planet använder vi både kartesiska koordinater (x, y) och polära koordinater (ρ, φ). Sambandet mellan dessa är

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

och vi skall nu översätta Laplaces ekvation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

till polära koordinater.

Vi har

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho} = \cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}$$

samt

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \varphi} = -\rho\sin\varphi\frac{\partial u}{\partial x} + \rho\cos\varphi\frac{\partial u}{\partial y}.$$

Alltså är

$$\begin{pmatrix} \frac{\partial u}{\partial \rho} \\ \frac{\partial u}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\rho\sin\varphi & \rho\cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}.$$

Invertering (matrisinvers) ger

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\frac{1}{\rho}\sin\varphi \\ \sin\varphi & \frac{1}{\rho}\cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \rho} \\ \frac{\partial u}{\partial \varphi} \end{pmatrix}.$$

Vi behöver nu

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\varphi\frac{\partial}{\partial \rho} - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial \varphi}\right)\left(\cos\varphi\frac{\partial u}{\partial \rho} - \frac{1}{\rho}\sin\varphi\frac{\partial u}{\partial \varphi}\right) =$$

$$= \cos^2\varphi\frac{\partial^2 u}{\partial \rho^2} - \sin\varphi\cos\varphi\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\right) - \frac{1}{\rho}\sin\varphi\frac{\partial}{\partial \varphi}\left(\cos\varphi\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\sin\varphi\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \varphi}\right) =$$

$$= \cos^2\varphi\frac{\partial^2 u}{\partial \rho^2} - \sin\varphi\cos\varphi\left(-\frac{1}{\rho^2}\frac{\partial u}{\partial \varphi} + \frac{1}{\rho}\frac{\partial^2 u}{\partial \rho\partial \varphi}\right) - \frac{1}{\rho}\sin\varphi\left(-\sin\varphi\frac{\partial u}{\partial \rho} + \cos\varphi\frac{\partial^2 u}{\partial \rho\partial \varphi}\right) +$$

$$+ \frac{1}{\rho^2}\sin\varphi\left(\cos\varphi\frac{\partial u}{\partial \varphi} + \sin\varphi\frac{\partial^2 u}{\partial \varphi^2}\right)$$

samt

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial u}{\partial\rho} + \frac{1}{\rho}\cos\varphi\frac{\partial u}{\partial\varphi}\right) =$$
$$= \sin^2\varphi\frac{\partial^2 u}{\partial\rho^2} + \sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial u}{\partial\varphi}\right) + \frac{1}{\rho}\cos\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial u}{\partial\varphi}\right) =$$
$$= \sin^2\varphi\frac{\partial^2 u}{\partial\rho^2} + \sin\varphi\cos\varphi\left(-\frac{1}{\rho^2}\frac{\partial u}{\partial\varphi} + \frac{1}{\rho}\frac{\partial^2 u}{\partial\rho\partial\varphi}\right) + \frac{1}{\rho}\cos\varphi\left(\cos\varphi\frac{\partial u}{\partial\rho} + \sin\varphi\frac{\partial^2 u}{\partial\rho\partial\varphi}\right) +$$
$$+ \frac{1}{\rho^2}\cos\varphi\left(-\sin\varphi\frac{\partial u}{\partial\varphi} + \cos\varphi\frac{\partial^2 u}{\partial\varphi^2}\right).$$

Sätt in dessa monster i Laplaces ekvation för att erhålla

$$\cos^2\varphi\frac{\partial^2 u}{\partial\rho^2} - \sin\varphi\cos\varphi\left(-\frac{1}{\rho^2}\frac{\partial u}{\partial\varphi} + \frac{1}{\rho}\frac{\partial^2 u}{\partial\rho\partial\varphi}\right) - \frac{1}{\rho}\sin\varphi\left(-\sin\varphi\frac{\partial u}{\partial\rho} + \cos\varphi\frac{\partial^2 u}{\partial\rho\partial\varphi}\right) +$$
$$+ \frac{1}{\rho^2}\sin\varphi\left(\cos\varphi\frac{\partial u}{\partial\varphi} + \sin\varphi\frac{\partial^2 u}{\partial\varphi^2}\right) + \sin^2\varphi\frac{\partial^2 u}{\partial\rho^2} +$$
$$+ \sin\varphi\cos\varphi\left(-\frac{1}{\rho^2}\frac{\partial u}{\partial\varphi} + \frac{1}{\rho}\frac{\partial^2 u}{\partial\rho\partial\varphi}\right) + \frac{1}{\rho}\cos\varphi\left(\cos\varphi\frac{\partial u}{\partial\rho} + \sin\varphi\frac{\partial^2 u}{\partial\rho\partial\varphi}\right) +$$
$$+ \frac{1}{\rho^2}\cos\varphi\left(-\sin\varphi\frac{\partial u}{\partial\varphi} + \cos\varphi\frac{\partial^2 u}{\partial\varphi^2}\right) = 0.$$

”Lite” förenkling ger då

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} = 0$$

som alltså är Laplaces ekvation i polära koordinater.